The Astrophysical Journal, 759:122 (5pp), 2012 November 10 doi:10.1088/0004-637X/759/2/122

BAYESIAN CONFIDENCE LIMITS OF ELECTRON SPECTRA OBTAINED THROUGH REGULARIZED INVERSION OF SOLAR HARD X-RAY SPECTRA

A. Gordon Emslie[1] and Anna Maria Massone[2]
[1] Department of Physics and Astronomy, Western Kentucky University, Bowling Green, KY 42101, USA; emslieg@wku.edu
[2] CNR-SPIN, Via Dodecaneso 33, I-16146 Genova, Italy; annamaria.massone@cnr.it
Received 2012 March 26; accepted 2012 September 16; published 2012 October 26

ABSTRACT

Many astrophysical observations are characterized by a single, non-repeatable measurement of a source brightness or intensity, from which we are to construct estimates for the true intensity and its uncertainty. For example, the hard X-ray count spectrum from transient events such as solar flares can be observed only once, and from this single spectrum one must determine the best estimate of the underlying source spectrum $I(\epsilon)$, and hence the form of the responsible electron spectrum $F(E)$. Including statistical uncertainties on the measured count spectrum yields a "confidence strip" that delineates the boundaries of electron spectra that are consistent with the observed photon spectrum. In this short article, we point out that the expectation values of the source brightness and its variance in a given photon energy bin are in general *not* (as has been assumed in prior works) equal to n, the number of counts observed in that energy bin. Rather, they depend both on n and on prior knowledge of the overall photon spectrum. Using Bayesian statistics, we provide an explicit procedure and formulas for determining the "confidence strip" (Bayesian credible region) for $F(E)$, thus providing rigorous bounds on the intensity and shape of the accelerated electron spectrum.

Key words: methods: data analysis – methods: statistical – Sun: flares – Sun: X-rays, gamma rays

1. INTRODUCTION

Astrophysics has been termed "the great uncontrolled experiment." In many astrophysical contexts, information is gleaned through independent, non-repeatable measurements of a count number n, which in general encompasses prescribed energy, direction, and time intervals (and, where applicable, polarization states). Each such observation, of course, represents but one sample drawn from a distribution of all the possible count numbers that *could* have been observed for the source in question, and from it we must derive expectation values for the actual source brightness μ and its variance σ^2.

A specific example refers to the use of solar flare hard X-ray spectral data to gain information on the electron spectrum that produces the hard X-rays through the bremsstrahlung process. As described in Brown et al. (2003), the mean source electron spectrum[3] $F(E)$ (electrons $\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{keV}^{-1}$) is found by solving the integral equation

$$I(\epsilon) = \frac{n\,V}{4\pi\,R^2}\int_{\epsilon}^{\infty} F(E)\,Q(\epsilon,E)\,dE, \qquad (1)$$

where $I(\epsilon)$ (photons $\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{keV}^{-1}$) is the observed photon spectrum, n and V are the mean number density (cm^{-3}) and volume (cm^3) of the source, R is 1 AU, and $Q(\epsilon, E)$ ($\mathrm{cm}^2\,\mathrm{keV}^{-1}$) is the electron–ion bremsstrahlung cross-section, differential in electron energy E.

As discussed by Piana et al. (2003), Equation (1) is ill-posed, so that the recovered form of $F(E)$ is critically dependent on the level of noise in the observed photon[4] spectrum $I(\epsilon)$. Since the measurements in question are not repeatable, the level of statistical noise (uncertainty) cannot be determined through repeated experiment. Therefore, to assess the effect of statistical counting errors on the recovered electron spectrum $F(E)$, a "confidence strip" of solutions is created by the following procedure. We start with the N count numbers $n_i = I(\epsilon_i)\,A_{\mathrm{det}}\,\Delta t\,\Delta\epsilon_i, i = 1,\ldots,N$ that constitute the observed spectrum (here A_{det} is the detector area (cm^2), Δt (s) is the observation time interval, and $\Delta\epsilon_i$ (keV) is the width of the ith energy bin) and to each we add a randomly generated noise value to create a *realization* of the count spectrum. Next, we calculate $F(E)$ for each such realization of the data through a regularized inversion procedure (Massone et al. 2003) based on Equation (1). The "river" of recovered solutions $F(E)$ defines the confidence strip that establishes acceptable bounds on the electron spectrum. As shown by Brown et al. (2006), the quality (both statistical significance and energy resolution) of the hard X-ray data from the *RHESSI* instrument (Lin et al. 2002), coupled with the development of robust methods (e.g., Johns & Lin 1992; Massone et al. 2003; Kontar et al. 2004) for solving Equation (1), results in the confidence strip being sufficiently narrow that "interesting" features in the electron spectrum $F(E)$ can be discerned.

[3] Brown et al. (2003) use the notation $\overline{n}$ and $\overline{F}(E)$ for the mean source density and electron spectrum, respectively. To avoid confusion with other notation in this paper, we will drop the bar superscripts here.

[4] Formally, there is a distinction between the number of photons emitted toward a detector and the number of counts observed in that detector; the relationship between the two depends on the detector response. In practice, this distinction may be nicely avoided by rewriting Equation (1) in terms of the count spectrum $I(q)$ and replacing the cross-section $Q(\epsilon, E)$ with the effective cross-section $K(q, E) = \int_q^{\infty} D(q,\epsilon)\,Q(\epsilon, E)\,d\epsilon$, where $D(q,\epsilon)$ is the detector response matrix—see Piana et al. (2007). Since this distinction is not critical to the goals of this paper, we do not choose to do this here, and the terms "photon" and "count" will be used interchangeably throughout.

The determination of the confidence strip requires a method through which to ascribe an appropriate value to the statistical noise at each measured data point. It is important to note that the observed count numbers n_i pertain to independent sets of photons in distinct energy bins, and so obey independent statistics. Therefore, estimates of the statistical noise can be obtained through the use of independent random sampling procedures at each data point.

Naively, for each data point i, the best estimate of the actual source brightness μ_i (in count units) is estimated by identifying it with n_i, the actual number of counts observed, so that the mean and variance of the parent Poisson distribution are both set to n_i (see, e.g., Bevington & Robinson 1992). Bevington & Robinson do explicitly recognize that the standard error should really be $\sqrt{\mu_i}$, where μ_i is the mean count number of the distribution of possible observations in that energy bin, but state (p. 40, using our notation) that "often we cannot make more than one measurement . . .. Thus, we are forced to use $\sqrt{n_i}$ as an estimate of the standard deviation of a single measurement."

While the identifications $\mu_i = n_i$ and $\sigma_i = \sqrt{n_i}$ are indeed valid for reasonably large count values $\gtrsim$20 (see below), they are *not* valid for small count numbers. To illustrate this with a simple example, suppose we observe *no* counts. Such a measurement is, of course, consistent with no source being present. However, it is also consistent with a source that has a non-zero brightness μ and which just happens to produce no counts in the particular observational interval concerned. Thus assigning a brightness expectation value $\mu_i = 0 \pm \sqrt{0}$ would clearly be incorrect.

Gehrels (1986) has provided analytic expressions for the upper (μ_u) and lower (μ_ℓ) limits of the value of μ appropriate to a measurement of n counts, as a function of the confidence level $C(S)$, where $C(S)$ is the integral of the normal distribution from $-\infty$ up to S standard errors from the mean.[5] These are

$$\sum_{x=0}^{n-1} \frac{\mu_\ell^x\, e^{-\mu_\ell}}{x!} = C; \quad \sum_{x=0}^{n} \frac{\mu_u^x\, e^{-\mu_u}}{x!} = 1 - C. \tag{2}$$

It should be noted that these equations are implicit formulas for μ_ℓ and μ_u. Gehrels (1986) has also provided explicit, but approximate, formulas (e.g., his Equations (7) and (11)). The one-standard-error ($S = 1$) approximate expressions are

$$\mu_\ell = n - \sqrt{n - \frac{1}{4}}; \quad \mu_u = n + 1 + \sqrt{n + \frac{3}{4}}. \tag{3}$$

For moderately large values $n \gtrsim 20$, these results are indeed closely approximated by the estimates $(\mu_\ell, \mu_u) = n \pm \sqrt{n}$. However, for low count numbers n, Equation (3) provides values for μ_ℓ and μ_u that differ significantly from these simple estimates.

In this article we use a Bayesian analysis (Section 2) to show how observations of a *set* of counts n_i in photon energy bins $(\epsilon_i, \epsilon_i + \Delta\epsilon_i)$ (i.e., a spectrum) can be used to produce, for each photon energy bin i, analytic expressions for the expectation value $\overline{\mu}_i$ of the source brightness and its standard error σ_i. It is then a straightforward matter to use the Bayesian prediction formula to generate a Poisson-based random realization $\widetilde{n}_i$ for the count in energy bin i. The ensemble of $\widetilde{n}_i$ values constitutes a single realization of the count spectrum, which can then (Section 3) be inverted using a regularized technique to generate a corresponding realization of the source electron spectrum $F(E)$. As we shall show, application of a Bayesian analysis, coupled with a priori knowledge of the possible forms of photon spectra $I(\epsilon)$ produced by bremsstrahlung emission (Brown & Emslie 1988), allows us to more accurately obtain the range of acceptable electron spectra $F(E)$.

[5] $C(1) = 0.8413$, $C(2) = 0.9772$, etc.

2. BAYESIAN ANALYSIS

2.1. Expectation Value of the Mean Brightness and Standard Error Associated with a Single Count Measurement

Define the conditional probability density $f(\mu|\, n)$ such that $f(\mu|\, n)\, d\mu$ is the probability that, given an observed count rate n, the actual mean source brightness (in count units) lies between μ and $\mu + d\mu$. A straightforward Bayesian analysis (e.g., Gelman et al. 1995) shows that

$$f(\mu|\, n) = \frac{P(n|\, \mu)\, g(\mu)}{\int_0^\infty P(n|\, \mu')\, g(\mu')\, d\mu'}, \tag{4}$$

where $P(n|\, \mu)$ is the (Poisson) probability of observing n counts from a source of mean brightness μ,

$$P(n|\, \mu) = \frac{e^{-\mu}\, \mu^n}{n!}, \tag{5}$$

and $g(\mu)$ is the assumed prior probability density for μ. It is common in Bayesian analyses (e.g., Cox & Hinkley 1974) to use conjugate functions to represent the prior and posterior distributions, and for the Poisson distribution the pertinent conjugate form is the gamma distribution (Gelman et al. 1995). We therefore take $g(\mu)$ to have a gamma distribution form, with hyperparameters α and β:

$$g(\mu) = G(\mu; \alpha, \beta) \equiv \frac{\beta^\alpha}{\Gamma(\alpha)}\, e^{-\beta\mu}\, \mu^{\alpha-1} \quad (\alpha > 0; \quad \beta > 0), \tag{6}$$

where $\Gamma(a)$ is the gamma function. Substituting Equations (5) and (6) in Equation (4) gives

$$f(\mu|\, n) = \frac{(1+\beta)^{n+\alpha}}{\Gamma(n+\alpha)}\, e^{-(1+\beta)\mu}\, \mu^{n+\alpha-1} \equiv G(\mu; n+\alpha, 1+\beta). \tag{7}$$

The observation n thus changes the distribution of μ values from its assumed prior form $G(\mu; \alpha, \beta)$ to the (conjugate) posterior form $G(\mu; n+\alpha, 1+\beta)$. Concomitantly, the Bayesian point estimators of the mean and variance change from their prior values

$$\overline{\mu}_p = \frac{\alpha}{\beta}\ , \qquad \sigma_p^2 = \frac{\alpha}{\beta^2} \tag{8}$$

to posterior values

$$\overline{\mu} = \frac{n+\alpha}{1+\beta}\ , \qquad \sigma^2 = \frac{n+\alpha}{(1+\beta)^2}. \tag{9}$$

For the uniform prior $g(\mu) =$ constant (i.e., $\alpha = 1, \beta = 0$), the posterior mean and variance are both shifted by unity ($\overline{\mu} = n + 1, \sigma^2 = n + 1$), reflecting the underlying skewness of the Poisson distribution. As a simple illustration of this case, consider the case $n = 0$ (no counts observed in a particular interval), for which $f(\mu\,|\,0) = G(\mu; 1, 1) = e^{-\mu}$. The probability density (per unit μ) that the source brightness μ actually *is* 0 is $f(0\,|\,0) = e^{-0} = 1$. Indeed, since $f(\mu\,|\,0)$ is a monotonically decreasing function of μ, this is also the most

likely scenario. However, it is not the *only* possible scenario: all positive values of μ are possible, albeit with monotonically decreasing probability densities. As a result, the expectation values of the brightness and its variance are *not* zero, but rather (from Equation (9) with $n = 0$, $\alpha = 1$, and $\beta = 0$) $\overline{\mu} = 1$ and $\sigma^2 = 1$. Thus, when no counts are observed, and a uniform prior is assumed, we should assign the source brightness and its standard error the value (1 ± 1).

2.2. Determination of the Prior Distribution Hyperparameters α and β for the Spectral Inversion Problem

In the bremsstrahlung process, photons of energy ϵ are produced by electrons of all energies $E \geqslant \epsilon$, and the form of the spectrum $I(\epsilon)$ is accordingly constrained (see, e.g., Brown & Emslie 1988). For example, for reasonable forms of the cross-section $Q(\epsilon, E)$ (Equation (1)), the photon spectrum $I(\epsilon)$ must be monotonically decreasing, no matter what the shape of the electron spectrum $F(E)$. Therefore, the values of n_i in the photon energy bins $(\epsilon_i, \epsilon_i + \Delta\epsilon_i)$, even though they represent observations of statistically independent photon sets, are not truly independent. The count values in each photon energy bin have to "fit in" to the overall spectral shape, and hence a uniform prior $g(\mu) =$ constant is manifestly not appropriate.

This insight provides a natural method for determining the prior means $\overline{\mu}_{p,i}$ and so the hyperparameter sets (α_i, β_i), $i = 1, \ldots, N$. Using the observed count spectrum $[\, n_i(\epsilon_i, \epsilon_i + \Delta\epsilon_i), i = 1, \ldots, N\,]$, we calculate the corresponding regularized (i.e., smoothed) electron spectrum $F^{(0)}(E)$. Substituting this smoothed electron spectrum in Equation (1), we generate a smooth photon spectrum $I(\epsilon)$ that, while still based on the original data, no longer suffers from the deviations associated with data noise. Discretization of this spectrum then provides the desired prior mean values $\overline{\mu}_{p,i}$ (and Poisson variances $\sigma_i^2 = \overline{\mu}_{p,i}$) for each photon energy bin $(\epsilon_i, \epsilon_i + \Delta\epsilon_i)$. Finally, the values of α_i and β_i for each energy bin follow from Equation (8), namely

$$\alpha_i = \frac{\overline{\mu}_{p,i}^2}{\sigma_i^2} = \overline{\mu}_{p,i}; \quad \beta_i = \frac{\overline{\mu}_{p,i}}{\sigma_i^2} = 1 \tag{10}$$

(see, e.g., Bolstad 2007). By Equations (9) and (10), the posterior estimates for $\overline{\mu}_i$ and σ_i are thus

$$\overline{\mu}_i = \frac{n_i + \overline{\mu}_{p,i}}{2}; \quad \sigma_i = \frac{\sqrt{n_i + \overline{\mu}_{p,i}}}{2}; \quad i = 1, \ldots, N. \tag{11}$$

At each point in the spectrum, the posterior point estimator of the intensity is simply the average of the prior mean $\overline{\mu}_{p,i}$ and the observed count n_i. Moreover, as a result of the information added by the observation n_i, the posterior estimate of the standard error is changed from its prior value $\sigma_{p,i} = \sqrt{\overline{\mu}_{p,i}}$. For observations n_i close to the prior $\overline{\mu}_{p,i}$, the standard error is reduced to a significantly smaller value; for example, an observation of $n_i = \overline{\mu}_{p,i}$ gives $\overline{\mu}_i = n_i$ and $\sigma_i = \sqrt{n_i/2}$: an observation n_i that is close to the prior point estimator $\overline{\mu}_{p,i}$ strengthens the accuracy of the mean $\overline{\mu}_i$ by a factor of $\sim\sqrt{2}$. On the other hand, an observation n_i that is far from the prior point estimator $\overline{\mu}_{p,i}$ results in a smaller reduction (and possibly even an increase) in the standard error; for example, a (highly unlikely) observation of $n_i = 3\,\overline{\mu}_{p,i}$ gives $\overline{\mu}_i = 2\overline{\mu}_{p,i} = 2n_i/3$; $\sigma_i = \sqrt{\overline{\mu}_{p,i}} = \sqrt{n_i/3}$; the posterior estimate of the mean is shifted significantly upward from the prior estimate, and the uncertainty in the posterior value is the same as for the prior value.

2.3. Comparison with Gehrels (1986)

The one-standard-error bounds on the posterior mean values are, by Equations (11),

$$\mu_{\ell,i} = \frac{n_i + \overline{\mu}_{p,i}}{2} - \frac{\sqrt{n_i + \overline{\mu}_{p,i}}}{2};$$
$$\mu_{u,i} = \frac{n_i + \overline{\mu}_{p,i}}{2} + \frac{\sqrt{n_i + \overline{\mu}_{p,i}}}{2}. \tag{12}$$

Comparing these with the results of Gehrels (1986)—Equations (3)—we see that the bounds from the Bayesian analysis are significantly tighter (by a factor of $\sim\sqrt{2}$) than the bounds derived by Gehrels (1986). This is a direct consequence of factoring in the prior values obtained from considerations of spectral continuity (see Section 2.2).

2.4. Posterior Count Distribution

With the form of the posterior distribution $f(\mu_i | n_i)$ for the source brightness μ_i in the ith energy bin determined, the Bayesian posterior predictive distribution

$$h(\widetilde{n}_i | n_i) = \int P(\widetilde{n}_i | \mu_i)\, f(\mu_i | n_i)\, d\mu_i \tag{13}$$

(e.g., Gelman et al. 1995, p. 8) may then be used to determine the probability distribution for the occurrence of a count value $\widetilde{n}_i$ given an actual observation n_i. Using Equation (5) for $P(\widetilde{n}_i | \mu_i)$ and Equation (7) for $f(\mu_i | n_i)$, we obtain the usual negative binomial distribution (e.g., Gelman et al. 1995, p. 53)

$$h(\widetilde{n}_i | n_i) = \binom{\alpha_i + n_i + \widetilde{n}_i - 1}{\widetilde{n}_i} \left(\frac{\beta_i + 1}{\beta_i + 2}\right)^{\alpha_i + n_i} \left(\frac{1}{\beta_i + 2}\right)^{\widetilde{n}_i}$$
$$\equiv \begin{cases} \dfrac{(1+\beta_i)^{n_i+\alpha_i}}{(2+\beta_i)^{\widetilde{n}_i+n_i+\alpha_i}} \dfrac{1}{\widetilde{n}_i\, B(\widetilde{n}_i, n_i + \alpha_i)} & ;\ \widetilde{n}_i \neq 0 \\ \left(\dfrac{1+\beta_i}{2+\beta_i}\right)^{n_i+\alpha_i} & ;\ \widetilde{n}_i = 0\,, \end{cases} \tag{14}$$

where $B(a, b) = \Gamma(a)\,\Gamma(b)/\Gamma(a + b)$ is the beta function.[6] For moderately large values of n_i, the distribution $h(\widetilde{n}_i | n_i)$ is very closely approximated by a normal distribution with mean and variance both equal to n_i; however, for low values of n_i, the distribution $h(\widetilde{n}_i | n_i)$ not only differs substantially from such a normal distribution, but also differs substantially from a Poisson distribution with mean n_i.

Two illustrative examples of the posterior predictive distribution (negative binomial) forms, compared to the corresponding Poisson distributions with means $\overline{\mu}_{p,i}$ and n_i, respectively, are shown in Figure 1. For low count rates (left panel of Figure 1), the Bayesian posterior predictive distribution of counts $h(\widetilde{n}_i | n_i)$ can differ significantly from the Poisson distributions using either the observed count rate n_i or the prior mean $\overline{\mu}_{p,i}$ as mean; for higher count rates (right panel of Figure 1), this distinction is less important.

3. APPLICATION TO DATA

3.1. Procedure

For the hard X-ray spectral inversion problem defined in Section 1, the analysis of the preceding section suggests the

[6] Note that $\sum_{\widetilde{n}_i=0}^{\infty} h(\widetilde{n}_i | n_i) = 1, \forall\, n_i, \alpha_i, \beta_i$.

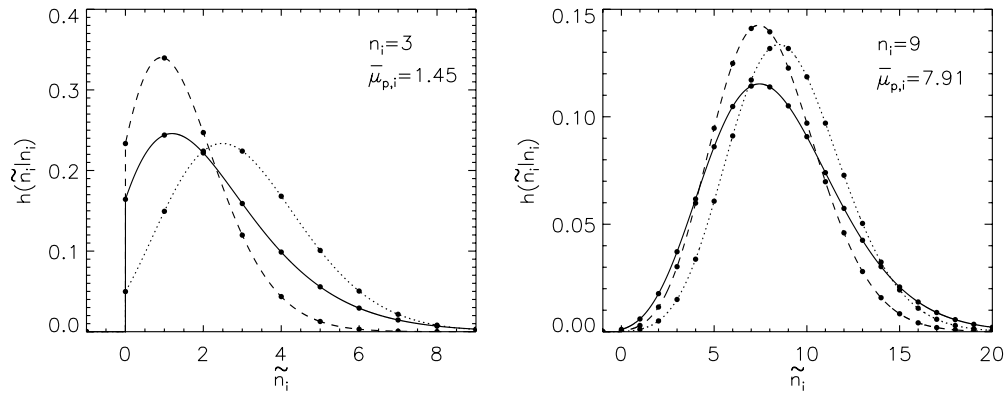

Figure 1. Solid curves: two examples of the posterior predictive count distributions $h(\widetilde{n}_i\,|\,n_i)$, for the n_i and $\overline{\mu}_{p,i}$ values shown in each panel. Values $\alpha_i = \overline{\mu}_{p,i}$ and $\beta_i = 1$ (see Equation (10)) have been used in both cases. The dots indicate the values of $h(\widetilde{n}_i\,|\,n_i)$ at the pertinent (integer) values of $\widetilde{n}_i$; the continuous curves illustrate the general shape of the distribution, to facilitate comparison with the other distributions depicted. Dashed curves: corresponding Poisson distributions $P(\widetilde{n}_i\,|\,\overline{\mu}_{p,i})$. Dotted curves: corresponding Poisson distributions $P(\widetilde{n}_i\,|\,n_i)$.

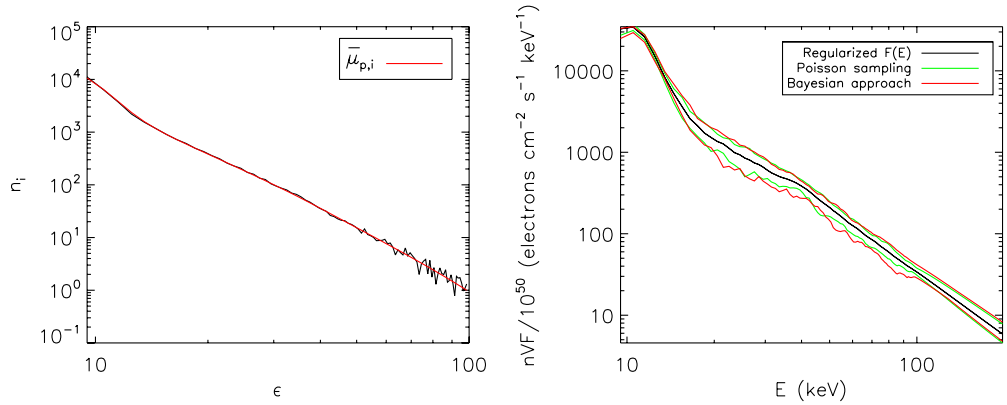

Figure 2. Left panel: count spectrum for a 2 s interval during the solar flare of 2002 April 14, 22:24:36 UT. Superimposed (red line) is the prior count spectrum $[\,\overline{\mu}_{p,i}, i = 1, \ldots, N\,]$ corresponding to the regularized electron spectrum $F^{(0)}(E)$ obtained from the data through Equation (1). Right panel, black line: regularized electron spectrum $F^{(0)}(E)$ corresponding to the original photon count data. The spectrum is weighted by nV, the number of target protons in the source—see Equation (1)—and is in units of 10^{50} electrons $\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{keV}^{-1}$. Right panel, green lines: the upper and lower boundaries of the confidence strip of electron spectra $F^{(k)}(E)$, $k = 1, \ldots, M$ based on a set of $M = 50$ count spectrum realizations. Each spectrum $F^{(k)}(E)$ was obtained by applying random noise at each photon spectrum data point, with the amount of noise determined through sampling of a Poisson distribution with a mean μ_i equal to the observed count rate n_i. Right panel, red lines: the upper and lower bounds of the confidence strip (Bayesian credible region), again based on $M = 50$ realizations of the observed count spectrum, but using the Bayesian procedure of Section 3.1.

following procedure for determination of the confidence strip of electron spectra $F(E)$ corresponding to an observed count spectrum $[\,n_i(\epsilon_i, \epsilon_i + \Delta\epsilon_i), i = 1, \ldots, N\,]$:

1. for the given count spectrum, determine the corresponding electron spectrum $F^{(0)}(E)$ through regularized inversion (Massone et al. 2003) of Equation (1);
2. use this electron spectrum $F^{(0)}(E)$ to obtain, by evaluating the integral in Equation (1), a best-estimate photon spectrum $I(\epsilon)$ and hence a prior mean count spectrum $[\,\overline{\mu}_{p,i}, i = 1, \ldots, N\,]$;
3. for each energy bin i, use the posterior predictive distribution (14), with the values $\alpha_i = \overline{\mu}_{p,i}$ and $\beta_i = 1$ (Equation (10)), to determine the probability distribution for the occurrence of a count value $\widetilde{n}_i$:

$$h(\widetilde{n}_i\,|\,n_i) = \begin{cases} \dfrac{2^{n_i+\overline{\mu}_{p,i}}}{3^{\widetilde{n}_i+n_i+\overline{\mu}_{p,i}}}\,\dfrac{1}{\widetilde{n}_i\,B(\widetilde{n}_i, n_i+\overline{\mu}_{p,i})} & ;\ \widetilde{n}_i \neq 0 \\ \left(\dfrac{2}{3}\right)^{n_i+\overline{\mu}_{p,i}} & ;\ \widetilde{n}_i = 0\,, \end{cases} \tag{15}$$

4. generate a set of M realizations of the count spectrum $\{\,[\,\widetilde{n}_i^{(k)}(\epsilon_i, \epsilon_i + \Delta\epsilon_i), i = 1, \ldots, N\,], k = 1, \ldots, M\,\}$, with each realization obtained by randomly sampling the probability distribution $h(\widetilde{n}_i\,|\,n_i)$ at each energy bin i;
5. from each realization of the count spectrum $[\,\widetilde{n}_i^{(k)}(\epsilon_i, \epsilon_i + \Delta\epsilon_i), i = 1, \ldots, N\,]$, construct a regularized electron spectrum $F^{(k)}(E)$ using the method of Massone et al. (2003);
6. combine the electron spectra $\{\,F^{(k)}(E), k = 1, \ldots, M\,\}$ to produce the confidence strip corresponding to the "Bayesian credible region" (e.g., Gelman et al. 1995) for the observed photon spectrum.

3.2. Example

We here illustrate the effect of employing the procedure of Section 3.1. We took the hard X-ray count spectrum observed by *RHESSI* (Lin et al. 2002) for the flare on 2002 April 14, 22:24:36 UT (left panel of Figure 2). This is a relatively weak flare with a moderately steep photon spectrum (spectral index $\gamma \simeq 4$); the intensity varies from $\sim$50 $\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{keV}^{-1}$ at 10 keV to $\sim$0.005 $\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{keV}^{-1}$ at 100 keV. A two-second integration time, with a *RHESSI* effective detector area of some 100 cm^2 and an energy bin size of 1 keV, thus gives a count number that ranges from $\sim 10^4$ at energies $\epsilon \simeq 10$ keV to ~ 1 at energies

$\epsilon \simeq 100$ keV. This count spectrum thus allows our method to be tested over a wide range of statistical accuracies.

We first added random noise to each point in the observed spectrum according to the naive (and previously ubiquitously employed—see, e.g., Piana et al. 2003) process of sampling a Poisson distribution with mean $\mu_i = n_i$. We then constructed the corresponding electron spectrum $F^{(k)}(E)$ for each such realization of the count spectrum by inverting Equation (1) using the method of Massone et al. (2003), with the smoothing parameter λ in the Tikhonov inversion procedure determined using the "3σ cumulative residuals" method (Piana et al. 2003). Repeating this process for $M = 50$ count spectrum realizations resulted in a confidence strip, for which the boundaries are depicted as green lines[7] in the right panel of Figure 2.

We then repeated the experiment using the Bayesian procedure of Section 3.1. We first calculated, using the method of Massone et al. (2003), the regularized electron spectrum $F^{(0)}(E)$ (black line in right panel of Figure 2) corresponding to the observed counts $[\, n_i(\epsilon_i, \epsilon_i + \Delta\epsilon_i),\, i = 1, \ldots, N\,]$ and then determined the corresponding (smoothed) prior photon spectrum $I(\epsilon)$ and its equivalent count spectrum (red line in left panel of Figure 2). Next, the posterior predictive distributions $[\, h(\widetilde{n}_i \,|\, n_i),\, i = 1, \ldots, N\,]$ were randomly sampled to generate $M = 50$ realizations of the photon data, each of which was then inverted to yield an electron spectrum $F^{(k)}(E)$. The upper and lower bounds of the confidence strip $\{F^{(k)}(E), k = 1, \ldots, M\}$ are shown in the right panel of Figure 2 as red lines.

Because the posterior predictive distribution (13) involves a convolution of the Poisson distribution for a given mean value μ with a distribution of such inferred mean values, the variance of the posterior predictive distribution is generally somewhat larger than the variance of the Poisson distribution kernel in Equation (13). Therefore, the confidence strips derived through the Bayesian approach are somewhat broader than those derived through the "naive" approach of sampling a set of Poisson distributions with means μ_i corresponding to the observed count values n_i. Comparing the red and green boundaries in the right panel of Figure 2 shows that this broadening is especially evident for the lower bound of the confidence strip.

To further study the results of the Bayesian confidence strip approach, we performed two additional experiments. First, we repeated the above analysis of the 2002 April 14 event using values for the smoothing parameter λ used in the Tikhonov regularized inversion algorithm (Massone et al. 2003) that were obtained using different criteria, including the Morozov discrepancy principle (Massone et al. 2003) and a 2σ-cumulative-residual criterion (cf. Piana et al. 2003). While the confidence strips thus obtained did, of course, vary with the value of λ, the *differences* between the confidence strips obtained using the "naive" method and those using the Bayesian approach were very consistent—specifically, the latter were somewhat wider, especially at the low-count-rate end of the spectrum. Second, we revisited the work of Brown et al. (2006), who investigated the ability of various methods to reconstruct synthetic source electron spectra characterized by various "interesting" short-wavelength features (e.g., cusps, notches) superimposed on a smooth background. Use of the Bayesian approach resulted in confidence strips that, while somewhat broader than those obtained by Brown et al. (2006), nevertheless still allowed the faithful identification of sharp features in the mean source electron spectrum, as well as its overall shape.

[7] Note that by using a "rectangular" form of the regularized inversion procedure (Kontar et al. 2004), and exploiting the fact that photons of a given energy ϵ are produced by all electrons with energy $E \geqslant \epsilon$, the range of E values over which each $F^{(k)}(E)$ can reliably be determined ($10\ \mathrm{keV} \lesssim E \lesssim 200\ \mathrm{keV}$) is greater than the range ($10\ \mathrm{keV} \lesssim \epsilon \lesssim 100\ \mathrm{keV}$) of observed photon energies ϵ.

4. CONCLUSIONS

We have demonstrated that, for a source that produces observed counts with a Poisson distribution and that happens to produce an observed count rate n in a single measurement, the expectation value and variance of the source brightness μ based on that single measurement are in general *not* (as has previously been assumed) equal to n. For the spectral inversion problem (1), the values of the prior means $\overline{\mu}_{p,i}$ in energy bins $(\epsilon_i, \epsilon_i + \Delta\epsilon_i)$ can be obtained through forward calculation of the photon spectrum $I(\epsilon)$ corresponding to the regularized electron spectrum $F^{(0)}(E)$ obtained from the data. Knowledge of these prior mean values then allows us to determine a set of Bayesian posterior predictions ("realizations") $\{[\, \widetilde{n}_i^{(k)}(\epsilon_i, \epsilon_i + \Delta\epsilon_i),\, i = i, \ldots, N\,];\, k = 1, \ldots, M\}$ of the count spectrum, each of which can then be inverted using regularization methods (Massone et al. 2003) to obtain one of a set of acceptable solutions $\{F^{(k)}(E);\, k = 1, \ldots, M\}$ for the electron spectrum.

Application of these results to the regularized spectral inversion of solar flare hard X-ray spectra (both observed and constructed from simulated data) shows that use of this procedure has a slight broadening effect on the confidence strip of solutions. However, this effect is not large enough to affect the ability of regularized spectral inversion methods to reveal both the overall shape of the mean source electron spectrum and the presence of localized spectral features in it. We therefore encourage use of the Bayesian methodology of Section 3.1 in the future analysis of solar hard X-ray spectra.

We thank Mark Payton, Professor of Statistics at Oklahoma State University, for his comments on various drafts of this manuscript and Brian Dennis for pointing out the existence of the Gehrels (1986) paper. A.G.E. was supported by NASA Grant NNX10AT78J; A.M.M. was supported by the European Community Framework Programme 7, "High Energy Solar Physics Data in Europe (HESPE)," Grant Agreement No. 263086.

REFERENCES

Bevington, P. R., & Robinson, D. K. 1992, Data Reduction and Error Analysis for the Physical Sciences (2nd ed.; New York: McGraw-Hill)
Bolstad, W. M. 2007, Introduction to Bayesian Statistics (Hoboken: Wiley)
Brown, J. C., & Emslie, A. G. 1988, ApJ, 331, 554
Brown, J. C., Emslie, A. G., Holman, G. D., et al. 2006, ApJ, 643, 523
Brown, J. C., Emslie, A. G., & Kontar, E. P. 2003, ApJ, 595, L115
Cox, D. R., & Hinkley, D. V. 1974, Theoretical Statistics (London: Chapman & Hall)
Gehrels, N. 1986, ApJ, 303, 336
Gelman, A., Carlin, J. B., Stern, H. S., & Rubin, D. B. 1995, Bayesian Data Analysis (London: Chapman & Hall)
Johns, C. M., & Lin, R. P. 1992, Sol. Phys., 137, 121
Kontar, E. P., Piana, M., Massone, A. M., Emslie, A. G., & Brown, J. C. 2004, Sol. Phys., 225, 293
Lin, R. P., Dennis, B. R., Hurford, G. J., et al. 2002, Sol. Phys., 210, 3
Massone, A. M., Piana, M., Conway, A., & Eves, B. 2003, A&A, 405, 325
Piana, M., Massone, A. M., Hurford, G. J., et al. 2007, ApJ, 665, 846
Piana, M., Massone, A. M., Kontar, E. P., et al. 2003, ApJ, 595, L127