FIST: a fast visualizer for fixed-frequency acoustic and electromagnetic inverse scattering problems

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Abstract

We describe a software package for the solution of two dimensional inverse scattering problems based on the linear sampling method. Our tool is designed according to an Object-Oriented paradigm and provides a rapid visualization of the objects profile from noisy far-field data in the case of both inhomogeneous and impenetrable objects. An application to microwave tomography is considered, showing that our tool can be used for both simulating hardware performances and testing the effectiveness of simulators for electromagnetic and acoustic computation.

Key words: Inverse scattering problems; linear sampling method

1 Introduction

Acoustic and electromagnetic inverse scattering problems [1] are often concerned with the restoration of the shape of some scattering object from complete or partial measurements of the scattered field corresponding to a fixed-frequency wave interrogation. Inverse scattering methods are involved in several applications, where essentially only indirect information about some physical parameters are known. A typical example is in microwave tomography, where cancerous tissues may be detected by means of non-invasive microwave interrogation [2].

Many of these methods are non-linear. As a result, they are difficult to solve and, very often, time consuming, the reason for these difficulties lying in both the numerical instability and non-linearity of the problem. Typical restoration methods belong to the class of Newton-like iterative schemes [1,3], which are characterized by notable accuracy but are both computationally expensive and slow to converge. A different approach is represented by the factorization methods which provide reconstructions of the scatterer profiles by requiring the solution of ill-conditioned linear systems at the points of a grid containing the scatterer. The main features of these methods are that they are more computational effective and that no *a priori* information about the object is necessary to perform the restoration (since no initialization procedure is required).

In the present paper we describe a software package based on one of these methods, the linear sampling method [4,5]. In particular, this method is used to solve inverse scattering problems in the case of both acoustic and electromagnetic two-dimensional scattering data. FIST (Fast Inverse Scattering Tool) has been developed according to an Object-Oriented approach and optimized in order to rapidly produce images of the scatterer from the far-field pattern. It can be applied for promptly verifying the effectiveness of acoustic and electromagnetic simulators computing the scattered near- or far-field (i.e., solving the direct problem), given the sources and the boundary conditions, or for simulating the performances of hardware configurations. Of course FIST is not only a simulator: after a systematic validation, it can be tested on experimental data sets and the reliability of this approach assessed in significant real-world applications.

The plan of the paper is as follows. In Section 2 we give a brief outline of the linear sampling method; in Section 3 we discuss the computational performances of our approach and in Section 4 the architecture and interface characteristics of FIST. Finally, in Section 5, we will consider an application concerning medical imaging.

2 The linear sampling method

Exhaustive formulations of the linear sampling method in the case of acoustic and electromagnetic scattering are given in [6] and [7], respectively. At the basis of the method is the far-field equation

$$\int_{\Omega} u_\infty(\hat{x}, d) g_z(d) ds(d) = \Phi_{\infty,z}(\hat{x}) \quad , \tag{1}$$

where $u_\infty(\hat{x}, d)$ is the far-field pattern of the scattered field corresponding to incident direction d, observation direction $\hat{x}$ and $g_z(d)$ is an indicator function in $L^2(\Omega)$, $\Omega = \{x \in \mathbb{R}^2, |x| = 1\}$. The right hand side $\Phi_{\infty,z}(\hat{x})$ is the far-field pattern of the fundamental solution $\Phi(x, z)$ of the two-dimensional Helmholtz equation, z being a generic point in $\mathbb{R}^2$ (in this scheme we consider the case of a single scatterer placed in a uniform background medium but generalizations to multiple scatterers and a background that is piecewise constant are straightforward [6,7]). In [8] it is proved that if z is a point approaching the boundary of the scatterer from the inside, there is an approximate solution of (1) whose L^2-norm blows up to infinity and, in fact, remains large outside the scatterer. This behaviour naturally inspires the following two-step algorithm for visualizing an object from its scattering data: given a grid containing the object, for each point of the grid:

- solve the linear system obtained by the discretization of the far-field equation;
- plot the euclidean norm of the solution.

The boundary of the scatterer will be given by those points where this norm is large. We note that the linear system solved in the first step is strongly ill-conditioned; therefore the solution must be obtained by means of some regularization algorithms. If

$$\mathbf{F}\mathbf{g} = \mathbf{\Phi} \tag{2}$$

is the discretized version of equation (1), then a possible approach [5] is to determine the one-parameter family $\{\mathbf{g}_\lambda\}_{\lambda>0}$ of solutions of the Tikhonov problem

$$\|\mathbf{F}\mathbf{g}_\lambda - \mathbf{\Phi}\|^2 + \lambda\|\mathbf{g}_\lambda\|^2 = \min \tag{3}$$

where $\|\cdot\|$ is the (canonical) Euclidean norm and the minimum is over the Euclidean solution space. The optimal value of the regularization parameter λ is fixed by means of the generalized discrepancy principle [9], i.e. by solving the equation

$$\|\mathbf{F}\mathbf{g}_\lambda - \mathbf{\Phi}\|^2 - h^2\|\mathbf{g}_\lambda\|^2 = 0 \tag{4}$$

where h is an estimate of the amount of noise affecting the measured far-field matrix $\mathbf{F}$. In [10] it is shown that the regularized solution of equation (2) can be effectively determined also by means of iterative regularization algorithms and that an extension of the generalized discrepancy principle to these algorithms, based on heuristic arguments, provides reliable results.

3 Computation

In the FIST implementation, the computation of the solution of both the Tikhonov minimization problem (3) and equation (4) is significantly simplified by initially computing the Singular Value Decomposition (SVD) of the matrix $\mathbf{F}$. Indeed, if p is the rank of $\mathbf{F}$ and $\mathbf{F}^*$ is its adjoint matrix, $\{\sigma_k, \mathbf{v}_k, \mathbf{u}_k\}_{k=1}^p$ is its singular system defined such that

$$\mathbf{F}\mathbf{v}_k = \sigma_k \mathbf{u}_k \quad ; \quad \mathbf{F}^*\mathbf{u}_k = \sigma_k \mathbf{v}_k \quad . \tag{5}$$

It is easily proved [11] that the Tikhonov regularized solution obtained by solving equation (3) is

$$\mathbf{g}_\lambda = \sum_{k=1}^{p} \frac{\sigma_k}{\sigma_k^2 + \lambda} (\mathbf{\Phi}, \mathbf{u}_k)\mathbf{v}_k \quad , \tag{6}$$

and therefore

$$\|\mathbf{g}_\lambda\|^2 = \sum_{k=1}^{p} \frac{\sigma_k^2}{(\sigma_k^2 + \lambda)^2} |(\mathbf{\Phi}, \mathbf{u}_k)|^2 \quad . \tag{7}$$

Furthermore, in terms of the singular system, equation (4) becomes

$$\sum_{k=1}^{p} \frac{\lambda^2 - h^2\sigma_k^2}{(\sigma_k^2 + \lambda)^2} |(\mathbf{\Phi}, \mathbf{u}_k)|^2 = 0 \quad . \tag{8}$$

Therefore our linear sampling method implementation is characterized by three computational steps:

step 1: computation of the SVD of the far-field matrix;
step 2: computation of the optimal value of the regularization parameter by solving (8);
step 3: computation of the norm of the optimal regularized solution by means of (7).

In order to evaluate the numerical complexity of the linear sampling method in this implementation, we consider a test experiment where a conducting kite (see Figure 1), with Dirichlet boundary conditions, is surrounded by N equispaced emitting/receiving antennas so that the far-field matrix has dimension $N \times N$. The first step (step 1) is made only once for each scattering experiment and has numerical complexity $O(N) = N^{3/2}$ [12]. The remaining two steps are performed for each point of the grid containing the scatterer and are super-linear (step 2) and linear (step 3), respectively. In particular, step 2 is computed by means of a zero-finding routine based on the Van Wijngaarden-Dekker-Brent method [13]. A comparison between the theoretical complexity $O(N^{3/2})$ and the experimental computational time in the case of the test problem is given in Figure 2 for increasing values of N. Finally, in Figure 3 we show the reconstruction of the kite given by FIST in the case of $N = 32$ with a 256×256 computational grid. The far-field pattern for this experiment has been computed by means of the Nyström method [1] and 1% gaussian noise has been added to each entry of the far-field matrix.

4 Architecture and interface

The FIST code has been implemented using C++ and follows an Object-Oriented paradigm. Presently, three classes have been developed, managing the input/output procedure (*Experiment Class*), the geometrical framework (*Grid Class*) and the numerical computation (*Resolutor Class*). In order to simplify the input step, the Experiment Class requires a standard input format given by a ".ffh" ASCII file containing all information about the scattering data (field of view, number of transmitting and receiving antennas, dimension of the data file) and a ".ffd" compressed file, containing the scattering data as double precision complex number strings. The Grid Class manages the initialization of the computational grid where the far-field equation is solved and allows the user to zoom into a refined mesh in those regions where more featured profiles require a more detailed computation of the indicator function. Finally, the Resolutor Class, which is the computational core of FIST, performs the SVD of the far-field matrix, provides the optimal value of the regularization parameter, computes the norm of the indicator function for each point of the grid and manages the application of edge-detection procedures in a post-processing step.

A simple graphical interface has been developed (see Figure 4) allowing the user to load the input files, set the reconstruction parameters (number of sampling points in the grid, field of view, number of iterations in the zero-finding routine) and save the reconstructed image as an ASCII or a standard image file. A post-processing step allows the user to

- refine the sampling of the computational grid, thus obtaining a zooming effect in those regions of the scatterer characterized by a more featured behaviour;
- segment the scattering profile by means of different edge-detection algorithms.

Several non-standard libraries have been included in the software in order to

Table 1
Electrical parameters for the Microwave Tomography experiment: ϵ is the electrical permittivity and σ is the electrical conductivity.

skin/fat	$\epsilon = 5.14$	$\sigma = 0.14$(S/m)
tumor	$\epsilon = 50.0$	$\sigma = 1.20$(S/m)

facilitate the development of a fast mathematical core, to build up the interface and to reduce possible memory leak problems.

5 An application to microwave tomography

From a clinical point of view, the main advantages of microwave tomography is its extremely mild invasivity, especially if compared to other structural imaging techniques like X-ray computerized tomography or functional procedures like PET or SPECT. Whatever image reconstruction method is adopted, microwave tomography sinograms will always provide low-resolution visualizations of the body under investigation, although, at least in principle, reliable restorations can be obtained even in the case of rather low contrasts. Here we consider the mammography experiment illustrated in Figure 5, where 36 emitting/receiving microwave antennas are uniformly placed around a disk of diameter 8 cm (the breast) containing a small disk of diameter 1 cm (the tumor) placed at 0.2 cm from the center. The incident field is characterized by a fixed wavenumber $k = 1$ cm^{-1} in the vacuum surrounding the breast, while the electrical permittivity and conductivity corresponding to the thin/fat part of the breast and to the tumor are given in Table 1. In Figure 5(a) we show the geometrical setup of this experiment. We applied FIST to the far-field pattern obtained by means of a finite element code based on [14] and 1% gaussian noise has been added to each entry of the far-field matrix. The scatterer has been embedded in a 256x256 computational grid. The result, obtained after around 4 seconds of computational time on our 1.5 GHz PC is represented in Figure 5(b) where a gray level is associated to each value of the norm of the regularized solution of the discretized far-field equation (here white corresponds to the lowest norm and black to the highest one). Figure 5(c) and Figure 5(d) are the result of the post-processing step, where an edge-detection procedure has been applied to the reconstruction. In particular, Figure 5(c) represents the discrete Laplacian of the image in Figure 5(b), computed by means of a Finite Difference method, while Figure 5(d) is the result of the application of the Canny algorithm for edge detection to the same image [15]. This last profile presents an evident artifacts near the boundary of the breast but clearly point out the position and the approximate dimension of the tumor.

6 Open problems

The modular design at the basis of FIST clearly favor the introduction of new applications by simple addition of further modules or functions. We are currently working at constructing a 3D Resolutor Class able to invert magnetic and electrical far-field patterns from fully inhomogeneous, anisotropic scattering experiments. Furthermore we would like to address the solution of the far-field equation by means of different regularization algorithms able to exploit *a priori* knowledges on the scatterer, for example by projections onto convex subsets of the solution space. We believe that this approach would produce a significant improvement of the spatial resolution of the method. Finally, a systematic validation of FIST is in due course, with the aim of assessing the reliability of this numerical approach in the case of different real-data applications.

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Fig. 1. The profile of the conducting kite used for testing the computational effectiveness of FIST.

Fig. 2. Numerical experiment evaluating the computational time for a test problem. The test problem is the one to reconstruct the profile of a conducting kite from far-field data recorded by an increasing number of equally spaced antennas. The experimental computational time is compared with the theoretical time $O(N^{3/2})$.

Fig. 3. FIST reconstruction of the conducting kite with Dirichlet boundary conditions. In this case the far-field matrix is 36×36, the computational grid is 256×256 and the wavenumber of the incident wave is $k = 1$.

Fig. 4. FIST interface

(a) (b)

(c) (d)

Fig. 5. The Microwave Tomography experiment and FIST reconstructions: (a) geometry of the simulation; (b) the norm of the regularized solution is plotted for each point of the computational grid; (c) application of the discrete Laplacian to the image in (b); (d) application of the Canny edge-detection algorithm.