Regularized reconstruction of the differential emission measure from solar flare hard X-ray spectra

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ABSTRACT

A numerical algorithm for reconstructing the differential emission measure from thermally interpreted bremsstrahlung photon spectra is introduced. The method is based on the Tikhonov regularization theory for solving ill-posed inverse problems and exploits 'ad hoc' boundary conditions to improve the restoration accuracy. A discussion of the applicability limitations of the method is performed for both simulated data and real photon spectra recorded by RHESSI.

Subject headings: Sun: flares

1. Introduction

Although it is well-established (Brown, 1971) that bremsstrahlung continuum radiation is the source for solar flare hard X-rays, it is currently unclear whether the electron distribution responsible of the emission comprises non-thermal particles trapped in a low density plasma (thin-target), particles 'injected' and 'stopped' in a dense plasma (thick-target) or a spatial distribution of locally Maxwellian electrons with a location-dependent temperature (or even some mixture of these three situations). In all cases, however, equation (Craig and Brwon, 1986)

$$J(\epsilon) = \int_V \int_\epsilon^\infty F(E, \mathbf{r}) n(\mathbf{r}) Q(\epsilon, E) dE dV \tag{1}$$

holds, where $J(\epsilon)$ is the resulting bremsstrahlung emission rate (photons per sec per unit photon energy ϵ), $n(\mathbf{r})$ is the plasma density and $F(E, \mathbf{r})$ is the electron flux density spectrum

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(electrons per unit area per sec per unit electron energy E) at location $\mathbf{r}$ in the source volume V via the bremsstrahlung cross-section $Q(\epsilon, E)$, differential in ϵ. Under a purely thermal interpretation of equation (1) (Brown, 1974), the electron distribution is assumed to be locally Maxwellian, i.e.

$$F(E, \mathbf{r}) = \frac{2^{3/2}}{(\pi m_e)^{1/2}} \frac{n(\mathbf{r})E}{[kT(\mathbf{r})]^{3/2}} e^{-E/kT(\mathbf{r})} \quad , \tag{2}$$

and the photon spectrum provides information on the differential emission measure (Craig and Brown, 1976, 1986)

$$\xi(T) = \int_{S_T} \frac{n^2(\mathbf{r})}{|\nabla T|} dS_T \quad , \tag{3}$$

differential in T, where S_T is a constant temperature surface.

The aim of the present paper is to discuss the problem of restoring numerically stable and physically significant forms for the differential emission measure given measurements of the photon spectrum. From the mathematical side, this inverse problem is particularly challenging, essentially because of the strong intrinsic numerical instability of the relation between $\xi(T)$ and $J(\epsilon)$. However, in the last years, two issues have made this inversion problem more intriguing than before. First, the fact that the RHESSI mission (Lin et al., 2002) can provide photon spectra with unprecedented spectral resolution and accuracy; second, the fact that the regularization theory for solving ill-posed inverse problems proved to be notably effective in the analysis of RHESSI spectra for the thin-target non-thermal case (Massone et al., 2003; Piana et al. 2003). This case, in fact, defines a standard mean source $\overline{F}(E)$ to which other models can be referred and therefore may be the best framework for studying the thermal problem.

The plan of the paper is as follows. In Section 2 we briefly discuss the thermal interpretation of the bremsstrahlung emission and describe the equations involved in this model. In Section 3 the ill-conditioning of the problem is pointed out and the regularization method adopted to reduce the numerical instability is introduced. In Section 4 we validate this regularization approach in the case of some significant synthetic cases. Finally Section 5 is devoted to the application of the method to real RHESSI photon spectra.

2. The thermal model

Purely thermal interpretations of X-ray emitted photon spectra have been trated in two regimes. In the first one, expression (2) for the local electron distribution is inserted into the model independent equation (1) and the the Kramers cross-section (Koch and Motz,

1959) is used for $Q(\epsilon, E)$. The result is an integral equation relating the photon spectrum and the differential emission measure, which has been treated by Piana et al. (1995) by means of regularization methods (Craig and Brown (1986) noticed that an analogous equation, i.e. the same kind of relation between $J(\epsilon)$ and $\xi(T)$, can be obtained starting from the Bethe-Heitler form for the cross-section, although with a further approximation in the computation). The main drawback of this approach is that it works under a coarse approximation on the cross-section, since it is well-established (Massone et al., 2004) that the 'true' cross-section in the bremsstrahlung interaction contains relativistic terms and has a much more complex nature than the Kramers' or Bethe-Heitler's approximations.

In the present paper we will follow the more general approach, whereby the isotropic source-averaged electron flux spectrum $\overline{F}(E)$, defined as

$$\overline{F}(E) = \frac{1}{\overline{n}V} \int_V n(\mathbf{r}) F(E, \mathbf{r}) dV \quad , \tag{4}$$

is first obtained from the photon spectrum by solving the equation

$$J(\epsilon) = \overline{n}V \int_\epsilon^\infty \overline{F}(E) Q(\epsilon, E) dE \quad , \tag{5}$$

using a precise form for $Q(\epsilon, E)$. Then the thermal distribution (2) is inserted into equation (4) and $\overline{F}(E)$ related to $\xi(T)$ by

$$\overline{F}(E) = \frac{1}{\overline{n}V} \frac{2^{3/2} E}{(\pi m_e)^{1/2}} \int_0^\infty \frac{\xi(T)}{(kT)^{3/2}} e^{-E/kT} dT \quad . \tag{6}$$

In this regime

- no assumption is made on the form of the cross-section. Equations (5), (6) are therefore the most general equations for a thermal interpretation of the bremsstrahlung solar X-rays emission;
- in order to obtain reconstructions of the emission measure function from measurements of the photon spectrum, two unstable successive inverse problems must be addressed: the first one, equation (5), leads to the mean flux electron spectrum $\overline{F}(E)$ from $J(\epsilon)$ and the second one, equation (6), leads to the emission measure $\xi(T)$ from $\overline{F}(E)$. Therefore this approach is more general from a physical point of view but more difficult to solve from a numerical point of view.

An important issue in this context is to establish the compatibility between the thermal interpretation of the data and the form of a spectral data. A formal answer to this problem is given in (Brown and Emslie, 1988), where it is shown that an electron spectrum is

compatible with a thermal interpretation if and only if it is 'completely monotonic', i.e. the first derivative is always negative, the second is always positive and so on. However, electron data are affected by noise and also numerical differentiation is an unstable inverse problem. It follows that the computation of only the first two or three order derivatives is reliable and data which do not fulfill this derivative test, because of the numerical uncertainty could still be consistent with a thermal interpretation.

3. Regularization and boundary conditions

In principle the problem of consistency between a mean electron spectrum and a thermal interpretation could be solved if a completely reliable inversion method were available: given the data vector, a thermal interpretation of it is possible if and only if the differential emission measure obtained by means of the inversion method does not have any negative components. We point out that a direct forward-fitting approach to solving equation (6) is ambiguous, due to the so-called ill-posedness of this kind of problems: indeed, very oscillatory $\xi(T)$, even with significant negative components, may fit the data vectors with the same accuracy as standard power laws or isothermal input functions.

Following Piana (1994), Piana et al. (2003) and Kontar et al. (2005), our idea is to address this problem by means of a numerical algorithm provided by the regularization theory for ill-posed inverse problems, although keeping in mind that the effectiveness of any inversion approach in this case is much weaker than for almost any other linear inverse problem. Indeed, from a quantitative point of view, it can be shown (Piana et al., 1995) that, for typical data parameters, the condition number associated with equations (6), i.e. its degree of numerical instability, is of the order of 10^{10}, which is much bigger than the condition number, around 10^3, associated with the non-thermal problem. In other words, given that a regularization algorithm essentially expresses the approximate solution as a smoothed linear combination of some basis functions, in the non-thermal problem $\overline{F}(E)$ can be obtained by using around ten basis functions in the case of data vector of around 100 points, while in the thermal problem only two or at most three basis functions can be used to expand $\xi(T)$. Therefore, in the recovery of $\xi(T)$ it is necessary to introduce more severe constraints than the one adopted in the $\overline{F}(E)$ inversion procedure described, for example, in (Piana et al., 2003). Even despite this, it is impossible to achieve a temperature resolution in some sense comparable with the spectral resolution with which $\overline{F}(E)$ can be reconstructed through the solution of the bremsstrahlung equation (5).

Measuring T in keV, adopting the change of variable $y = 1/T$ and neglecting all the

constants, equation (6) becomes

$$\overline{F}(E) = E \int_0^\infty f(y) \exp(-Ey) dy \quad , \tag{7}$$

where $f(y)$ is defined as

$$f(y) = \frac{\xi(1/y)}{y^{1/2}} \quad . \tag{8}$$

Equation (7) gives a continuous representation of the model, while astronomical observations provide only sampled and noisy versions of the spectra. In other terms we need to solve the linear inverse problem with discrete data

$$g_n = (Lf)_n \qquad n = 1, \ldots, N \quad , \tag{9}$$

where

$$g_n = \overline{F}(E_n) \qquad n = 1, \ldots, N \tag{10}$$

and

$$(Lf)_n = E_n \int_0^\infty f(y) \exp(-E_n y) dy \qquad n = 1, \ldots, N \quad . \tag{11}$$

The Tikhonov method for solving equation (9) looks for stable approximations of $f(y)$ by addressing the minimum problem

$$\|Lf - \mathbf{g}\|^2 + \lambda \|Df\|^2 = \text{minimum} \tag{12}$$

where D is an appropriate differential operator. $D = I$ (zero-order regularization), which was effective for the non-thermal problem (Piana et al., 2003), does not assure here a sufficient degree of smoothness, while regularization orders higher than one are less advantageous for computational reasons.

On the other hand, let us consider the minimum problem (12) when D is the first derivative operator. A straightforward computation shows that the solution of such equation can be analytically found simply by introducing the two boundary conditions (Piana et al., 1995)

$$f(0) = 0 \tag{13}$$

and

$$\lim_{y \to \infty} f'(y) = 0 \quad . \tag{14}$$

This approach is particularly advantageous for two reasons. First, keeping the solution in a functional space (in fact, no discretization in the T space is made) avoids introducing truncation errors which would further increase the numerical instability of the computation. Second, the boundary condition (13) constrains the solution to behave well at $y = 0$, thus

improving the restoration accuracy at high T. However, adopting this condition must be compatible with the behaviour of the data vector. In order to study such compatibility we considered asymptotic properties of the Laplace Transform of a function, described by the so-called Abelian theorems (Widder, 1946). We found that condition (13) is compatible with the electron spectrum when the electron spectral index δ is bigger than zero and therefore the corresponding photon spectral index is bigger than 1. It must be observed that condition (14), introduced for technical reasons, is basically unphysical and, for some experimental conditions, may lead to artifacts in the small T region.

4. Applications

In the present section we want to validate the effectiveness of the regularization approach introduced in the previous section. In particular, we will show that this method represents a reasonable alternative to the derivative test to assess the compatibility between the data and its thermal interpretation. Then the temperature resolution achievable by the method will be discussed and a more realistic form of $\overline{F}(E)$ will be considered for the inversion.

4.1. Compatibility test

As discussed in the Introduction, the availability of a reliable inversion method allows to determine whether a photon spectrum can be interpreted according to a purely thermal description of the emission. As an example, let us consider the case of a photon spectrum produced by an averaged electron spectrum described by

$$\overline{F}(E) = \begin{cases} \frac{1}{E^2} & E \geq E_c \\ 0 & E < E_c \end{cases} , \tag{15}$$

where E_c represents an intermediate energy cut-off. Equation (15) is discretized according to a uniform sampling starting from a minimum sampled energy E_{min} and the resulting data vector is affected by (realistic) poissonian noise. In Figure 1 we applied the Tikhonov inversion method for three possible experimental situations determined by the values of the pair E_c, E_{min}. In Figure 1(a) and (b), $E_c = E_{min}$, i.e. the problem reduces to the reconstruction of a power-law. The result is clearly encouraging: a stable differential emission measure is restored and its spectral index is close to the theoretical value. In Figure 1(c), $E_{min} < E_c$, i.e. the electron spectrum is given by some (but very few) points equal to zero with the other points sampling the power law. This time the reconstruction is characterized

by significant negative components and is absolutely inaccurate. This behaviour is coherent with the fact that, in accordance with the derivative test, a non-monotonic electron spectrum is not compatible with a purely thermal interpretation of the emission.

4.2. Temperature resolution

From a theoretical point of view, given an electron data set, the temperature resolution achievable from this inversion method can be determined by looking at the behaviour of the last significant basis function in the expansion of the regularized solution. If, formally,

$$\xi_\lambda(T) = \sum_{k=1}^{\infty} c_k^{(\mathbf{g},\lambda)} u_k(T) \tag{16}$$

is such expansion, substantially only few $u_k(T)$, say $\overline{k}$, actually contribute to $\xi_\lambda(T)$. It can be proven (Bertero et al., 1982) that $u_{\overline{k}}(T)$ has $\overline{k}-1$ zeros and therefore at each temperature range the achievable T resolution is given by the distance between two adjacent zeros at that specific range. For the reconstruction of δ-functions, only ? basis functions are used and the resolution in the range $[1, 10]$ keV is therefore approximately $\Delta =?$. Numerically, in Figure 2 we show how much we can approach two δ-functions in order that they are still visible in the reconstruction. The first δ-function is kept fixed at $T_0 = 2$ keV while the second one is moved from $T_0 =?$ (dotted in Figure 2(a)), $T_0 =?$ (dashed) and $T_0 =?$ (dot-dashed). $\overline{F}(E)$ for these three δ-function pairs has been computed and affected by poissonian noise and finally inverted using the regularization method. The results in figures 2(b), (c) and (d) show that the δ-function pair is still visible for the temperature distance $\Delta T =?$, but no more visible for $\Delta T =?$ in the case $T_0 =?$ but no more visible for $T_0 =?$.

Another way to assess the resolution power of the method is to consider the reconstruction of a top-hat function T, with increasing values of the right limit T_B of the top-hat temperature interval (the left limit T_A is kept fixed at $T = 2$ keV; see Figure 3(a)). The idea is to evaluate for which value of the right limit a significant part of the top-hat function (say 80%) is reconstructed by the method. Figures 3(b), (c), (d) show that this restoration power is obtained beginning from $T_B = \ldots$.

4.3. More realistic spectra

An averaged electron flux spectrum is typically characterized by a thermal or multi-thermal component at low electron energies plus a power-law behaviour at high electron

energies. A simple example is given by the simulated electron spectrum (Figure 4(a))

$$\overline{F}(E) = T_0^{-3/2} E e^{-\frac{E}{T_0}} + C \left(\frac{E}{50}\right)^{-\delta} \quad , \tag{17}$$

corresponding to the theoretical differential emission measure

$$\xi(T) = \delta(T - T_0) + C T^{-\delta+0.5} \quad , \tag{18}$$

where $T_0 = 4$ keV, $\delta = 1.1$ and C is an appropriate constant. Equation (17) is discretized according to a uniform sampling of 1 keV interval from $E_{min} = 10$ keV to $E_{max} = 150$ keV and the resulting vector is contaminated by poissonian noise. We inverted the low energy part of the spectrum up to $E = 50$ keV and the resulting differential emission measure is plotted in Figure 4(b), where the confidence limit has been obtained by repeating the inversion for different realization of the noise on $\overline{F}(E)$. The recovered $\xi(T)$ peaks at $T_0 \simeq 3.5$ keV, i.e. the influence of the power-law part of the spectrum at small energies results in a temperature shift.

5. RHESSI data

We perform a thermal interpretation of photon flux spectra recorded by RHESSI and apply the first-order Tikhonov regularization method with boundary conditions in order to restore the corresponding differential emission measure. As a first example, we consider a data set in the time interval around the emission peak of the July 23, 2002 flare. This vector is inverted by using zero-order regularization according to the procedure described in (Piana et al., 2003). The resulting averaged electron spectrum is represented in Figure 5 (a). The low energy part of this data vector is used as the input data in equation (6) and the corresponding differential emission measure is reconstructed by using the first-order Tikhonov algorithm. We fixed $E_{min} = 10$ keV and increased E_{max} up to ? keV, that is the energy up to where the reconstructed $\xi(T)$ is in accordance with the typical peaked behaviour. The reconstructed $\xi(T)$ is represented in Figure 5 (b) together with its confidence strip. This regularized differential emission measure is used in (7), (8) for $E > E_{max}$, in order to compute the thermal contribution $\overline{F}_{th}(E)$ to the electron spectrum. The purely non-thermal contribution to the spectrum is given by $\overline{F}(E) - \overline{F}_{th}(E)$, $E > E_{max}$ and represented in Figure 5 (c).

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Fig. 1.— Averaged electron flux spectrum used in equation (7) to restore $\xi(T)$. The analytical form is given by equation (21), with $T_0 = 4$ keV, $\delta = 1.1$ and the added noise is poissonian.

Fig. 2.— Averaged electron flux spectrum used in equation (7) to restore $\xi(T)$. The analytical form is given by equation (21), with $T_0 = 4$ keV, $\delta = 1.1$ and the added noise is poissonian.

Fig. 3.— Averaged electron flux spectrum used in equation (7) to restore $\xi(T)$. The analytical form is given by equation (21), with $T_0 = 4$ keV, $\delta = 1.1$ and the added noise is poissonian.

Fig. 4.— Averaged electron flux spectrum used in equation (7) to restore $\xi(T)$. The analytical form is given by equation (21), with $T_0 = 4$ keV, $\delta = 1.1$ and the added noise is poissonian.

Fig. 5.— Averaged electron flux spectrum used in equation (7) to restore $\xi(T)$. The analytical form is given by equation (21), with $T_0 = 4$ keV, $\delta = 1.1$ and the added noise is poissonian.